New Aspects of Electromagnetic Information Theory for Wireless and Antenna Systems

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***Abstract*—This paper investigates information-theoretic characterization, via Shannon's information capacity and number of degrees of freedom, of wave radiation (antenna) and wireless propagation systems. Specifically, the paper derives, from the fundamental physical point of view of Maxwell's equations describing electromagnetic fields, the Shannon information capacity of space-time wireless channels formed by electromagnetic sources and receivers in a known background medium. The theory is developed first for the case of sources working at a fixed frequency (time-harmonic case) and is expanded later to the more general case of temporally bandlimited systems (time-domain fields). In the bandlimited case we consider separately the two cases of time-limited and essentially bandlimited systems and of purely bandlimited systems. The developments take into account the physical radiated power constraint in addition to a constraint in the source L^2 norm which acts to avoid antenna superdirectivity. Based on such radiated power and current L^2 norm constraints we derive the Shannon information capacity of canonical wireless and antenna systems in free space, for a given additive Gaussian noise level, as well as an associated number of degrees of freedom resulting from such capacity calculations. The derived results also illustrate, from a new information-theoretic point of view, the transition from near to far fields.**

***Index Terms*—Antenna, degrees of freedom, electromagnetic information, information capacity, wave information, wireless.**

I. Introduction

THIS research is concerned with the formulation of fundamental wireless communication and antenna engineering problems at the crossroads of the well-established fields of electromagnetic theory and information theory. The interdisciplinary field constituted by such wave- and information-theoretic problems and their solutions can be descriptively termed *electromagnetic information theory* or, within the antenna focus, *antenna information theory*. Work in this area, combining wave physics with information theory, has a long history, dating back to the origins of information theory and, in particular, to the pioneering work on *light and information* by Gabor [1]–[3] who studied the number of degrees of freedom (NDF) [4] of diffraction-limited optical imaging systems via "essential dimension" considerations (e.g., "resolution cell" concepts). The NDF of imaging systems was investigated around the same time also by di Francia [5]–[8] who explored the role of the evanescent plane wave spectra in superresolution, as well as rigorous formal methodologies for NDF determination such as eigenvalue decompositions of integral operators and prolate spheroidal wave-function theory [9], [10]. Related work from the same period is that of Linfoot [11] and Fellgett and Linfoot [12] who literally treated imaging systems as communication channels. After these seminal investigations, further progress on NDF of wave radiation, propagation, and scattering systems has been reported from time to time by other authors [4], [13]–[18].

In recent years, the need for both fundamental theory and computational methodology for estimating both NDF and Shannon's information capacity [19], [20] of a variety of wave radiation and propagation systems, including random media channels, has been growing steadily, mostly due to its fundamental importance in space-time wireless channels including multiple-input multiple-output (MIMO) systems [21]–[23]. The present paper builds from recent work with this motivation, being of particular relevance the investigations carried out by: Hanlen *et al.* [24], [25] who investigated general communication channels within a general abstract operator theory in Hilbert spaces and, within the scalar framework, the information capacity in the formally tractable case of sources and fields in spherical regions under the standard source L^2 norm constraint; Hui *et al.* [26] who similarly provided general expressions for the information capacity of Gaussian wireless links from a spatial region to another; Hanlen and Fu [27] who proposed numerical techniques for estimating the NDF of general wave propagation systems including media formed by collections of scatterers embedded to a given background medium (e.g., free space); Jensen and Wallace [28] (see also their related past work in [29] and [30]) who studied the information capacity of Gaussian wireless links between two volumes under a constraint in the radiated power and who also proposed an approach to limit antenna superdirectivity while making the optimal power allocations for the different wave propagative modes connecting the volumes; Morris *et al.* [31] who also studied the effect of the superdirectivity on the information capacity of MIMO systems under a radiated power constraint; Gustafsson and Nordebo [32], [33] who studied the spectral efficiency of an antenna inside a sphere, as measured by the information capacity, by means of a model of MIMO channels combining information theory, antenna theory, and broadband matching theory; Migliore [34] who considered the capacity and NDF of a group of antennas and scatterers inside a sphere by means of the multipole expansion and equivalent electrical networks and who also proposed a way to take into account the temporal resources by means of a sampling theory in space and time [35]; Chakraborty and Franceschetti [36] who also considered the capacity of space-time communication channels by means of a generalized sampling theorem applied

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to space and time; Poon *et al.* [37] who considered NDF and information capacity of indoor wave propagation systems based on combined analytical and empirical, measurement-based models, while also illustrating, among other aspects, the role of cooperative versus noncooperative users in the environment; and Xu and Janaswamy [38] who proposed a noise-dependent definition to select the NDF in the approach in [18] and numerically studied the effect of multiple scattering in the propagation environment.

The overall goal of the present paper is the investigation of certain fundamental aspects not fully covered by previous work in this fruitful area, mostly on information-theoretic characterization, via Shannon's information capacity and NDF, of wave radiation (antenna) and wireless propagation systems. Specifically, the paper derives, from the fundamental physical point of view of Maxwell's equations describing electromagnetic fields, the information capacity of *space-limited, time-limited, and essentially bandlimited* wireless channels as well as *space-limited and strictly bandlimited* wireless channels formed by electromagnetic sources and receivers in a known background medium, under the assumption of additive Gaussian noise perturbations. While particular attention is given to free space, the results are based on Green's functions so that the general procedure and results derived in the paper apply to more general known media. Among other aspects, the derived information-theoretic results are also used to address related questions such as NDF for time-domain fields, the transition from near to far fields in the time domain, both of which are topics of much importance under the broader umbrella of time-domain electromagnetics, as well as the effect of finite transmit/receive times on the capacity and NDF (in the time-limited and essentially bandlimited case).

Methodologically, the electromagnetic information results are discussed first for time-harmonic excitation in Section III of the paper, and generalized later to more general time-domain fields in Section IV. The connection between the general electromagnetic information theory in Section IV, applicable to quite general broadband fields, and the time-harmonic theory in Section III (from which narrowband information-theoretic concepts such as spectral efficiency can be derived) is also discussed, along with illustration of the broader applicability of the general approach in Section IV. The formulation in Section IV builds from the one-dimensional time-domain treatment of (temporally) bandlimited channels by Gallager [39], which we generalize to the four-dimensional space-time domain in the particular physical context of electromagnetic wave radiation and propagation systems. Our work on the space-limited, time-limited and essentially bandlimited case appears to be new, while our work on the strictly bandlimited case is a rigorous alternative that complements related research in [35], [36] which is based on sampling theorems in space and time.

Particular attention is given in this paper to the fundamental problem of sources confined within a given spherical volume of radius a, subjected to upper bounds limiting the maximal allowed radiated power, and/or the maximal source L^2 norm or functional energy (related to impressed current levels in an antenna, and thereby also to the associated ohmic losses which must be bounded in practice as well as to superdirectivity control [31]), for a given noise level at the field receiver. Thus, the derived capacity and NDF results are immediately applicable to sources and receivers in free space that are supported in concentric spherical regions. However, many of the derived formalities have applicability to more general geometries, as follows from our discussion in Sections II-A and B of the paper where essentially the general index α in Section II-B will in general substitute the particular multipole-mode index $\alpha = j, l, m$ emphasized later in the paper (Section II-C and the rest of the paper). An important aspect of the electromagnetic information theory reported here is that our capacity and NDF calculations take into account practical restrictions in *both* the source L^2 norm and the radiated power. This is in contrast to all past work we are aware of where the main emphasis has been to either constrain the source L^2 norm [24], [25], [37] or, more recently, to constrain the radiated power only, while *indirectly* limiting the source L^2 norm by means of different approaches like the restriction of the transmission currents to specific subspaces [31], the use of channel models that include the effect of the losses and thermal noise [28], or the truncation of the number of terms in the expansions of the field [32], [33]. In fact, our adoption of this methodology complements recently reported approaches in [28] and [31] to estimate wireless electromagnetic capacity while penalizing antenna superdirectivity, and our treatment also provides a different and independent reformulation of some of the results in [28], [31] where, like those papers, we show that the radiated power alone is generally insufficient as a constraint for electromagnetic capacity calculations and that, instead, it must be complemented in practice by other physically motivated constraints (such as the L^2 norm constraint). Our treatment complements the analysis in those papers by considering the two most typically adopted constraints simultaneously, and also goes beyond the scope of that past work by extending the results to broadband fields. Also, regarding the calculation of the NDF, instead of assuming an asymptotic behavior of the functions used in the analysis in order to find a finite NDF or truncating the dimensionality *a priori* by other criteria (as has been done in past work in this area [4], [17], [18], [32]–[34], [37], [40]), we show that this is not necessary since the optimal power assignment maximizing the mutual information between transmitter and receiver already determines the number of channels that are useful for a given noise level.

II. Electromagnetic Information Channels in General Geometries

This section reviews the representation of general and particular spatial electromagnetic channels in terms of the singular system ($\mathbf{u}_\alpha$, $\mathbf{v}_\alpha$, σ_α; $\alpha = 1, 2, \ldots$) of the corresponding source-to-field mapping (see, e.g., [17] and [41, pp. 234–239]). This representation is then used in the rest of the paper to characterize information capacity and NDF.

A. General Sources and Fields in Hilbert Space

Consider an electromagnetic source $\mathcal{J}(\mathbf{r}, t)$ (a current density) confined to a volume V_T. The frequency domain representation of this source is obtained by taking the temporal Fourier transform of $\mathcal{J}(\mathbf{r},t)$, which gives the spectral current density

$$\mathbf{J}(\mathbf{r},f)=\int\mathcal{J}(\mathbf{r},t)e^{-\imath 2\pi f t}dt$$

where $\imath=\sqrt{-1}$. In the remainder of this section and in the next one, the focus will be on spatial electromagnetic information for a given frequency f so that for notational simplicity we will suppress the f dependence, e.g., we will use $\mathbf{J}(\mathbf{r})$ in place of $\mathbf{J}(\mathbf{r},f)$, with the understanding that all quantities depend on the frequency f. This source $\mathbf{J}(\mathbf{r})$ is assumed to belong to the Hilbert space $\mathcal{X}=L^2(V_T)$ of square integrable functions of support V_T with inner product defined by

$$\langle\mathbf{J}|\mathbf{J}'\rangle_{\mathcal{X}}\equiv\int_{V_T}\mathbf{J}^*(\mathbf{r})\cdot\mathbf{J}'(\mathbf{r})d^3r$$

where $*$ over a quantity means its complex conjugate.

The electric field $\mathbf{E}(\mathbf{r})$ is measured in a volume V_R. This *measured* field belongs to the Hilbert space $\mathcal{Y}=L^2(V_R)$ with inner product defined by

$$\langle\mathbf{E}|\mathbf{E}'\rangle_{\mathcal{Y}}\equiv\int_{V_R}\mathbf{E}^*(\mathbf{r})\cdot\mathbf{E}'(\mathbf{r})d^3r.$$

B. Singular System Representation of the Source-to-Field Mapping

The measured field $\mathbf{E}$ is given by

$$\mathbf{E}(\mathbf{r})=(P\mathbf{J})(\mathbf{r})=I_{V_R}(\mathbf{r})\int_{V_T}\mathbf{G}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')d^3r' \qquad (1)$$

where we have introduced the radiation operator $P:\mathcal{X}\to\mathcal{Y}$, $I_{V_R}(\mathbf{r})$ is an indicator function parameterized by V_R with value 1 if $\mathbf{r}\in V_R$ or 0 if $\mathbf{r}\notin V_R$, and $\mathbf{G}(\mathbf{r},\mathbf{r}')$ represents the dyadic Green function satisfying

$$\nabla\times\nabla\times\mathbf{G}(\mathbf{r},\mathbf{r}')-k^2\mathbf{G}(\mathbf{r},\mathbf{r}')=\imath 2\pi f\mu_0\mathbf{I}\delta(\mathbf{r}-\mathbf{r}')$$

where $\mathbf{I}$ is the identity dyadic, δ is Dirac's delta function, and $k=2\pi f\sqrt{\mu_0\varepsilon_0}$ is the wavenumber where ε_0 is the free-space permittivity and μ_0 is the free-space permeability. For finite and disjoint transmission and reception volumes, the Green function kernel in (1) is square integrable [4], [18], [41], [42] so that the operator P is compact and can be represented via the singular value decomposition (SVD) [41]–[43].

The SVD of P can be obtained from the eigendecomposition of the associated self-adjoint operator [41, pp. 237,238], [42, pp. 258,259]$PP^\dagger:\mathcal{Y}\to\mathcal{Y}$ where $P^\dagger:\mathcal{Y}\to\mathcal{X}$ represents the adjoint of P defined by $\langle P\mathbf{J}|\mathbf{E}\rangle_{\mathcal{Y}}=\langle\mathbf{J}|P^\dagger\mathbf{E}\rangle_{\mathcal{X}}$ so that

$$\left(P^\dagger\mathbf{E}\right)(\mathbf{r})=I_{V_T}(\mathbf{r})\int_{V_R}\mathbf{G}^*(\mathbf{r}',\mathbf{r})\cdot\mathbf{E}(\mathbf{r}')d^3r'. \qquad (2)$$

Let $\mathbf{u}_\alpha(\mathbf{r})$ be an eigenfunction of the operator $PP^\dagger$ (and receiver-side singular function of P), then

$$\left(PP^\dagger\mathbf{u}_\alpha\right)(\mathbf{r})=(\sigma_\alpha)^2\mathbf{u}_\alpha(\mathbf{r}) \qquad (3)$$

where σ_α is the respective singular value. The corresponding transmitter-side singular function is given by [41]

$$\mathbf{v}_\alpha(\mathbf{r})=\frac{\left(P^\dagger\mathbf{u}_\alpha\right)(\mathbf{r})}{\sigma_\alpha}. \qquad (4)$$

Expanding the kernel in (1) in terms of the singular functions so that

$$I_{V_R}(\mathbf{r})\mathbf{G}(\mathbf{r},\mathbf{r}')I_{V_T}(\mathbf{r}')=\sum_\alpha\sigma_\alpha\mathbf{u}_\alpha(\mathbf{r})\mathbf{v}_\alpha^*(\mathbf{r}') \qquad (5)$$

and defining

$$a_\alpha=\langle\mathbf{u}_\alpha|\mathbf{E}\rangle_{\mathcal{Y}} \qquad (6)$$

and

$$b_\alpha=\langle\mathbf{v}_\alpha|\mathbf{J}\rangle_{\mathcal{X}}, \qquad (7)$$

yields the alternative (singular system or diagonalizing) representation of (1)

$$a_\alpha=\sigma_\alpha b_\alpha. \qquad (8)$$

C. SVD for a Spherical Scanning Geometry in Free Space

Now consider the particular case of a spherical scanning geometry in free space where the source $\mathbf{J}(\mathbf{r})$ is confined to the spherical volume $V_T=\{\mathbf{r}\in R^3:r\equiv|\mathbf{r}|\le a\}$ and where the electric field $\mathbf{E}(\mathbf{r})$ is measured in a spherical scanning surface $V_R=\{\mathbf{r}\in R^3:r=b\}$ of radius b concentric to the source. This geometry is of interest for antennas (e.g., one can envision a spherical near-field scanning setup to test and compare antennas informationally) and enables the analytical calculation of the singular system of P by means of spherical wavefunction expansions.

1) The Multipole Expansion: In free space, the temporal Fourier transform of the measured field is given from the multipole expansion [44], [45] by

$$\mathbf{E}(\mathbf{r})=I_{V_R}(\mathbf{r})\sum_{j,l,m}\breve{a}_{j,l,m}\mathbf{A}_{j,l,m}(\mathbf{r}) \qquad (9)$$

where $\sum_{j,l,m}\equiv\sum_{j=1}^{2}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}$, where

$$\mathbf{A}_{1,l,m}(\mathbf{r})=\nabla\times\left[h_l^+(kr)\mathbf{Y}_{l,m}(\hat{\mathbf{r}})\right] \qquad (10)$$

and

$$\mathbf{A}_{2,l,m}(\mathbf{r})=\imath k h_l^+(kr)\mathbf{Y}_{l,m}(\hat{\mathbf{r}}) \qquad (11)$$

are the electric and magnetic multipole fields, respectively, where h_l^+ is the spherical Hankel function of the first kind (corresponding to outgoing waves) and order l, and $\mathbf{Y}_{l,m}$ is the vector spherical harmonic of degree l and order m defined by [46], [47]

$$\mathbf{Y}_{l,m}(\hat{\mathbf{r}})=\mathbf{L}Y_{l,m}(\hat{\mathbf{r}}) \qquad (12)$$

where $\mathbf{L} = -\imath\mathbf{r} \times \boldsymbol{\nabla}$ (the angular momentum operator), $\hat{\mathbf{r}} = (\theta, \phi)$ represents the direction of $\mathbf{r}$, and $Y_{l,m}(\hat{\mathbf{r}})$ represents the spherical harmonics [48, p. 787]. The expansion coefficients $\breve{a}_{j,l,m}$ in (9) are given by

$$\breve{a}_{j,l,m} = \langle \mathbf{B}_{j,l,m} | \mathbf{J} \rangle_{\mathcal{X}} \tag{13}$$

where the spherical wave functions $\mathbf{B}_{j,l,m}$ are defined by

$$\mathbf{B}_{1,l,m}(\mathbf{r}) = -\frac{\eta}{l(l+1)} I_{V_T}(\mathbf{r}) \boldsymbol{\nabla} \times [j_l(kr)\mathbf{Y}_{l,m}(\hat{\mathbf{r}})] \tag{14}$$

and

$$\mathbf{B}_{2,l,m}(\mathbf{r}) = -\frac{ik\eta}{l(l+1)} I_{V_T}(\mathbf{r}) j_l(kr) \mathbf{Y}_{l,m}(\hat{\mathbf{r}}) \tag{15}$$

where η is the free-space impedance (≈ 377 ohms), and j_l is the spherical Bessel function of order l.

The spherical wave functions $\mathbf{B}_{1,l,m}$ and $\mathbf{B}_{2,l,m}$ obey the orthogonality relations [see [45, Eq. (17)]]

$$\langle \mathbf{B}_{j,l,m} | \mathbf{B}_{j',l'm'} \rangle_{\mathcal{X}} = \|\mathbf{B}_{j,l,m}\|^2_{\mathcal{X}} \, \delta_{j,j'} \delta_{l,l'} \delta_{m,m'} \tag{16}$$

where here and henceforth $\|\cdot\|$ denotes Euclidean norm (so that $\|\mathbf{B}_{j,l,m}\|^2_{\mathcal{X}} = \langle \mathbf{B}_{j,l,m} | \mathbf{B}_{j,l,m} \rangle_{\mathcal{X}}$), where $\delta_{\cdot,\cdot}$ denotes the Kronecker delta and where, see (17) at the bottom of the page.

Similarly, it is not hard to show by borrowing from [49, Eqs. (5), (13), (16)], including the orthogonality property of the vector functions $\hat{\mathbf{r}} Y_{l,m}(\hat{\mathbf{r}})$, $\mathbf{Y}_{l,m}(\hat{\mathbf{r}})$, and $\hat{\mathbf{r}} \times \mathbf{Y}_{l,m}(\hat{\mathbf{r}})$ [50, pp. 1898–1901] as well as the recurrence relations for h_l^+, that the multipole fields, $\mathbf{A}_{1,l,m}$ and $\mathbf{A}_{2,l,m}$, are also orthogonal in the sense that

$$\langle \mathbf{A}_{j,l,m} | \mathbf{A}_{j',l'm'} \rangle_{\mathcal{Y}} = \|\mathbf{A}_{j,l,m}\|^2_{\mathcal{Y}} \, \delta_{j,j'} \delta_{l,l'} \delta_{m,m'} \tag{18}$$

where [see (19) at the bottom of the page] and

$$\|\mathbf{A}_{2,l,m}\|^2_{\mathcal{Y}} = l(l+1) k^2 b^2 \left| h_l^+(kb) \right|^2 . \tag{20}$$

2) Singular System Representation: Using (9) and (13), the kernel of P in (1) is found to be given by (5) with α substituted by j, l, m and

$$\mathbf{u}_{j,l,m}(\mathbf{r}) = I_{V_R}(\mathbf{r}) \frac{\mathbf{A}_{j,l,m}(\mathbf{r})}{\|\mathbf{A}_{j,l,m}\|_{\mathcal{Y}}} \tag{21}$$

$$\mathbf{v}_{j,l,m}(\mathbf{r}) = I_{V_T}(\mathbf{r}) \frac{\mathbf{B}_{j,l,m}(\mathbf{r})}{\|\mathbf{B}_{j,l,m}\|_{\mathcal{X}}} \tag{22}$$

and

$$\sigma_l^{(j)} = \|\mathbf{B}_{j,l,m}\|_{\mathcal{X}} \, \|\mathbf{A}_{j,l,m}\|_{\mathcal{Y}} \, . \tag{23}$$

The results above hold for both near and far fields. In the special far-field case, one can use the large-argument approximation $h_l^+(x) \approx (-\imath)^{l+1}/x e^{\imath x}$ [48, Eq. (11.160a)] to arrive from (19) and (20) at the approximation $\|\mathbf{A}_{j,l,m}\|^2_{\mathcal{Y}} \approx l(l+1)$ which in turn yields the far-field singular values

$$\left[\sigma_l^{(j)}\right]^2 = l(l+1) \, \|\mathbf{B}_{j,l,m}\|^2_{\mathcal{X}} \, . \tag{24}$$

The far-field singular values in (24) are well known to decay rapidly for $l \gtrsim ka$ [4], [43], [45], [51]–[53]. Since, as discussed in (9), j takes values 1 and 2, while m takes integer values from $-l$ to l, for each index l there are $2(2l+1)$ singular values so that an estimate of the NDF or *information content* [4] of the far field is $\mathrm{NDF} \approx 2ka(ka+2)$. On the other hand, this rule of thumb does not hold in the near zone, where we must use the general result (23).

Replacing the index $\alpha \to j, l, m$ in (8) yields the representation

$$\begin{aligned} a_{j,l,m} = \sigma_l^{(j)} b_{j,l,m} \quad &\forall\, j = 1,2; \;\; l = 1,2,\ldots \\ m &= -l, -l+1, \ldots, l-1, l. \end{aligned} \tag{25}$$

It is important to emphasize that the j, l, m indexes represent spatial modes that are physically differentiable (separable) due to the orthogonality of the functions $\mathbf{u}_{j,l,m}$ and $\mathbf{v}_{j,l,m}$ [via (21) and (22) or, alternatively, (16) and (18)] so they correspond to an effective number of transmitter and receiver modes that are potentially communicable between the antenna or scatterer of spatial dimension a and a receiver outside the antenna. Also note that (without further information) in view of (7) and (8) only the scalar quantities $b_{j,l,m}$ associated to a given source can be known about that source, i.e., only certain projections of the source onto the space of functions $\mathbf{u}_{j,l,m}(\mathbf{r})$, in particular, can be measured from the received signals. Hence, we will formulate next the communication problem in terms of the quantity $b_{j,l,m}$ as containing all the information about the transmitter state (as determined by the source $\mathbf{J}(\mathbf{r})$) that is in principle recoverable

$$\begin{aligned} \|\mathbf{B}_{1,l,m}\|^2_{\mathcal{X}} &= \frac{1}{2l+1} \left\{ (l-1) \, \|\mathbf{B}_{2,l-1,m}\|^2_{\mathcal{X}} + (l+2) \, \|\mathbf{B}_{2,l+1,m}\|^2_{\mathcal{X}} \right\} \\ \|\mathbf{B}_{2,l,m}\|^2_{\mathcal{X}} &= \frac{(\eta k)^2 a^3}{2l(l+1)} [j_l^2(ka) - j_{l-1}(ka) j_{l+1}(ka)]. \end{aligned} \tag{17}$$

$$\|\mathbf{A}_{1,l,m}\|^2_{\mathcal{Y}} = l^2(l+1)^2 \left| h_l^+(kb) \right|^2 + l(l+1) \left| 2h_l^+(kb) + \frac{kb}{2l+1} \left(l h_{l-1}^+(kb) - (l+1) h_{l+1}^+(kb) \right) \right|^2 \tag{19}$$

from the measurement of the receiver signal. No further information or components of this source can be estimated from the exterior field measurements alone, so the communication problem reduces to the mapping dictated (in singular system representation) by (25).

III. Space Electromagnetic Information Capacity

In this section the discrete representation of (1) in terms of an infinite number of spatial modes in (25) is interpreted as a set of parallel communication channels where each output $a_{j,l,m}$ is related to the input $b_{j,l,m}$ probabilistically due to the presence of noise (to be added in the following). The input modes $b_{j,l,m}$ represent random variables containing the information to be transmitted, therefore, the source $\mathbf{J}(\mathbf{r})$ is characterized by a set of random states or currents that are to be transmitted to the output $\mathbf{E}(\mathbf{r})$ in the presence of noise.

A. Noise Modeling

In practice, errors are introduced to the measurements mainly from imperfect antenna location, distortion of the field due to the test equipment, imprecise measurements of the fields, and numerical approximations [54]. Since most errors are independent and additive [54], a natural choice for the noise model is that of a white Gaussian random process $\mathbf{n}(\mathbf{r})$ defined by the usual property that for any $L^2(V_R)$ function $\mathbf{f}(\mathbf{r})$ the projection $\langle \mathbf{f}|\mathbf{n}\rangle$ is a zero mean Gaussian random variable with variance N_0 [39, p. 365]. In particular, if $\mathbf{n}(\mathbf{r})$ is expanded in terms of a set of orthonormal functions then the resulting set of coefficients are independent [39, p. 367]. Thus, adding this noise to the formulation in Section II (where $\mathbf{E}$ becomes $\mathbf{E}+\mathbf{n}$) one readily finds that the channel model in (8) takes the revised form

$$\begin{aligned} a_{j,l,m} &= \sigma_l^{(j)} b_{j,l,m} + n_{j,l,m} \quad \forall\, j=1,2 \\ l &= 1,2,\ldots; \quad m=-l,-l+1,\ldots,l-1,l. \end{aligned} \tag{26}$$

where

$$n_{j,l,m} = \langle \mathbf{u}_{j,l,m}|\mathbf{n}\rangle_{\mathcal{Y}} \tag{27}$$

and $E\left[n^*_{j,l,m} n_{j',l',m'}\right] = N_0 \delta_{j,j'}\delta_{l,l'}\delta_{m,m'}$ [V^2] where V indicates volts and $E[\cdot]$ indicates the expected value.

In some situations it may be convenient to modify the noise to allow different variances for different channels so that $E\left[n^*_{j,l,m} n_{j',l',m'}\right] = N_{j,l,m}\delta_{j,j'}\delta_{l,l'}\delta_{m,m'}$. The developments that follow still apply in this case after the replacement $N_0 \to N_{j,l,m}$. Also, for the case of colored noise where the set of coefficients $n_{j,l,m}$ have an arbitrary correlation matrix, an alternative procedure involving the eigendecomposition of the correlation matrix of the noise can be employed as described in [20, Ch. 9].

B. Capacity Constraints and Connection to Physical Quantities

One of the most important and common constraints for capacity calculations is the bounding of the square of the L^2 norm of the source which bounds the current levels and, therefore, the associated ohmic losses and the level of superdirectivity of the antenna [31]. This quantity is commonly known as the (functional) *energy* of the source whether it is directly related to the physical energy or not [41, p. 98]. For a deterministic source $\mathbf{J}(\mathbf{r})$ it is given by

$$\mathcal{E} = \|\mathbf{J}\|^2_{\mathcal{X}} = \int_{V_T} |\mathbf{J}(\mathbf{r})|^2\, d^3r = \sum_{j,l,m} |b_{j,l,m}|^2 \left[\frac{\mathrm{A}^2}{\mathrm{m}}\right] \tag{28}$$

where A indicates amperes and m meters. Bounding this quantity restricts the sources so that they are square integrable. At the same time, this bound restricts the amount of power radiated by the antenna since integral operators with square integrable kernels, as in the present radiation analysis, are bounded [41]. An insightful way of visualizing this in the framework of linear inversion theory is as follows. Having a bounded source L^2 norm requires satisfying the Picard condition $\sum_{j,l,m} |a_{j,l,m}|^2 / \left[\sigma_l^{(j)}\right]^2 < \infty$ [41, Eq. (10.96)]. However, $\infty > \sum_{j,l,m} |a_{j,l,m}|^2 / \left[\sigma_l^{(j)}\right]^2 > 1/\sigma^2_{\max} \sum_{j,l,m} |a_{j,l,m}|^2$ where $\sigma_{\max}$ denotes the largest of the singular values $\sigma_l^{(j)}$, so that the norm of the field (at the spherical surface) is also bounded. By referring to (31) where it is shown that the radiated power is equal to the L^2 norm of the far field (apart from a constant factor) we conclude that the radiated power is also bounded.

Mathematically, the source L^2 norm is the electromagnetic counterpart of the bounded power constraint of temporal channel communication theory (see, e.g., [39, Ch. 8]) but for antennas it does not represent the average radiated power which is calculated from Poynting's theorem [55, Eq. (4.21)] and is generally given in terms of the multipole moments by [51, Eq. (4.24)]

$$P = \frac{1}{2\eta} \sum_{j,l,m} l(l+1)\, |\breve{a}_{j,l,m}|^2 \tag{29}$$

$$= \frac{1}{2\eta} \sum_{j,l,m} l(l+1)\, \|\mathbf{B}_{j,l,m}\|^2_{\mathcal{X}}\, |b_{j,l,m}|^2 \tag{30}$$

$$= \frac{1}{2\eta} \sum_{j,l,m} \left[\hat{\sigma}_l^{(j)}\right]^2 |b_{j,l,m}|^2\, [\mathrm{W}] \tag{31}$$

where here and henceforth $\left[\hat{\sigma}_l^{(j)}\right]^2$ denotes the far-field singular values in (24) and W indicates watts. Note, however, that this constraint alone is not enough to find realistic solutions since bounding the radiated power will not necessarily lead to a bounded source L^2 norm in (28) (due to the decaying behavior of the singular values since $\hat{\sigma}_l^{(j)} \to 0$ as $l \to \infty$; see, e.g., [43, p. 41] and [41, p. 252]).

C. Space Capacity With Multiple Constraints

Consider the parallel channels in (26) where $n_{j,l,m}$ represents white Gaussian noise with variance N_0 and where the L^2 norm of the source is constrained to $\mathcal{E}$, i.e.

$$\sum_{j,l,m} E\left[|b_{j,l,m}|^2\right] = \sum_{j,l,m} \mathcal{E}_{j,l,m} \le \mathcal{E} \tag{32}$$

where $\mathcal{E}_{j,l,m} \equiv E\left[|b_{j,l,m}|^2\right]$ and the radiated power is constrained to P, i.e.

$$\frac{1}{2\eta}\sum_{j,l,m}\left[\hat{\sigma}_l^{(j)}\right]^2\mathcal{E}_{j,l,m} - P \leq 0 \tag{33}$$

where

$$\mathcal{E}_{j,l,m} \geq 0. \tag{34}$$

The capacity is obtained from maximizing the mutual information [20] between the input $b_{j,l,m}$ and the output $a_{j,l,m}$ by finding the optimal probability distribution for the set $\{b_{j,l,m}\}$. As is well known [19], [20], for uncorrelated Gaussian noise the mutual information is maximized when the input probability distribution is an uncorrelated Gaussian distribution with variance $\mathcal{E}_{j,l,m}$. Thus, the capacity is obtained by solving the optimization problem [56, p. 182]

$$\max_{\mathcal{E}_{j,l,m}}\sum_{j,l,m}\log_2\left(1+\frac{\mathcal{E}_{j,l,m}\left[\sigma_l^{(j)}\right]^2}{N_0}\right) \tag{35}$$

subject to (32)–(34).

A dual optimization problem associated to this capacity calculation can be found by means of the Lagrange multipliers technique [57, p.176] where the Lagrangian function is given by

$$\begin{aligned}\mathcal{L} = &-\sum_{j,l,m}\log_2\left(1+\frac{\mathcal{E}_{j,l,m}\left[\sigma_l^{(j)}\right]^2}{N_0}\right)\\ &+\lambda_1\left(\sum_{j,l,m}\mathcal{E}_{j,l,m}-\mathcal{E}\right)\\ &+\lambda_2\left(\frac{1}{2\eta}\sum_{j,l,m}\left[\hat{\sigma}_l^{(j)}\right]^2\mathcal{E}_{j,l,m}-P\right)\end{aligned} \tag{36}$$

and the necessary and sufficient conditions for optimality are given by the Kuhn-Tucker theorem [57, p. 138]

$$-\frac{\log_2 e}{\frac{N_0}{\left[\sigma_l^{(j)}\right]^2}+\mathcal{E}_{j,l,m}}+\lambda_1+\lambda_2\frac{1}{2\eta}\left[\hat{\sigma}_l^{(j)}\right]^2 = 0 \tag{37}$$

$$\sum_{j,l,m}\mathcal{E}_{j,l,m}-\mathcal{E} \leq 0 \tag{38}$$

$$\frac{1}{2\eta}\sum_{j,l,m}\left[\hat{\sigma}_l^{(j)}\right]^2\mathcal{E}_{j,l,m}-P \leq 0 \tag{39}$$

$$\lambda_1 \geq 0 \tag{40}$$

$$\lambda_2 \geq 0 \tag{41}$$

$$\lambda_1\left(\sum_{j,l,m}\mathcal{E}_{j,l,m}-\mathcal{E}\right) = 0 \tag{42}$$

$$\lambda_2\left(\frac{1}{2\eta}\sum_{j,l,m}\left[\hat{\sigma}_l^{(j)}\right]^2\mathcal{E}_{j,l,m}-P\right) = 0. \tag{43}$$

Without loss of generality from this point forward we will include the factor $\log_2 e$ in the Lagrange multipliers so that $\lambda_1 \to \lambda_1/\log_2 e$ and $\lambda_2 \to \lambda_2/\log_2 e$.

From (37) the optimal source L^2 norm assignment is

$$\hat{\mathcal{E}}_{j,l,m} = \left(\frac{1}{\lambda_1+\lambda_2\frac{1}{2\eta}\left[\hat{\sigma}_l^{(j)}\right]^2}-\frac{N_0}{\left[\sigma_l^{(j)}\right]^2}\right)^+ \tag{44}$$

where $(\cdot)^+ \equiv \max(0,\cdot)$. Replacing $\hat{\mathcal{E}}_{j,l,m}$ in (35) yields the space capacity with constraints in the source L^2 norm and radiated power

$$C = \sum_{j,l,m}\left[\log_2\frac{\left[\sigma_l^{(j)}\right]^2}{N_0}\left(\frac{1}{\lambda_1+\lambda_2\frac{1}{2\eta}\left[\hat{\sigma}_l^{(j)}\right]^2}\right)\right]^+ \tag{45}$$

where the multipliers λ_1 and λ_2 are chosen to agree with conditions (38)–(43). Special cases and illustrations of these general results will be given later, but next we wish to make a few remarks on four key themes that are of much interest to us, as both application and motivation aspects of this effort, and that apply to both the present space-information analysis as well as the more general broadband-spatio–temporal information analysis of Section III-C-I. These aspects are: 1) information-theoretic estimation of NDF of electromagnetic radiation and propagation channels; 2) antenna superdirectivity control via L^2 norm constraint; 3) near-to-far-field transition from the point of view of Shannon information; and 4) applicability to both transmit and receive configurations (informational reciprocity).

1) Number of Degrees of Freedom: In this paper, the NDF is defined as the number of channels with L^2 norm $\mathcal{E}_{j,l,m}$ in (44) greater than zero. The values of the Lagrange multipliers λ_1 and λ_2 that minimize the Lagrangian in (36) assign the available source L^2 norm $\mathcal{E}$ to the infinite channels in an optimal way, i.e., depending on the channel gains defined by the singular values. Because of the finite amount of resources, only a finite number of channels, out of the infinite number originally available, are used. This number (of effective wave modes that can be communicated) depends on the constraint level of the source L^2 norm $\mathcal{E}$, on the constraint in the radiated power P, on the noise variance N_0, and on the behavior of the singular values which in turn depends on the geometry of the system. Our analysis in (44) and associated expressions shows that it is not necessary to artificially truncate the number of channels *a priori*, as has been done, however, in the majority of past works in this area [4], [17], [18], [34], [36], [37], [40]. Instead, the finite number of channels appears naturally as a byproduct of maximizing the capacity. This definition of the NDF differs from the commonly used one in linear inverse problems where the NDF is associated to the number of "significant" singular values. This last definition is particularly appropriate for the case of rank deficient operators where the corresponding singular values have a step-like behavior with a clear gap between large and small singular values. For example, as follows from our discussion in (24), for large antennas and observation regions in the far zone of the antenna the NDF ($\approx 2ka(ka+2)$) is relatively well defined and independent of the noise level (see, e.g., [37], [58]). In

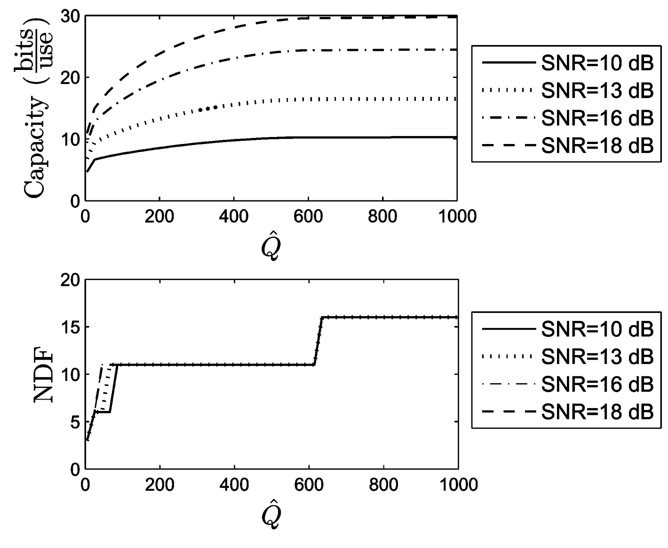

Fig. 1. Capacity and NDF as a function of the ratio $\hat{Q} = \mathcal{E}/P$ for a fixed power $P = 1\ \mathrm{mW}$ and for different levels of the SNR for a source with $ka = 0.38$ and concentric spherical receiver with $b = 2\ \mathrm{m}$.

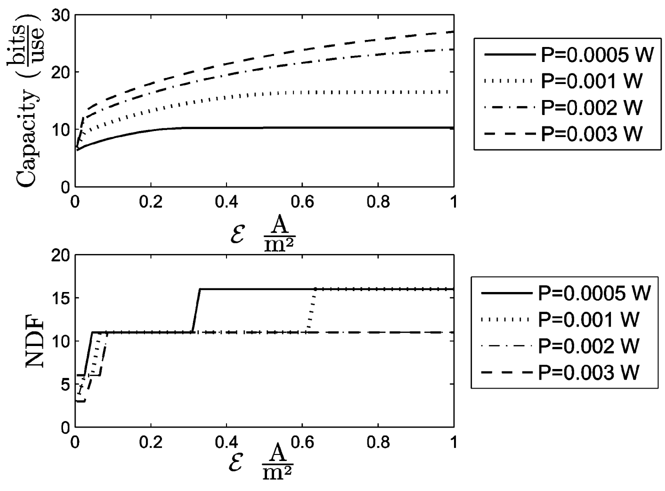

Fig. 2. Capacity and NDF as a function of the source L^2 norm bound $\mathcal{E}$ for a fixed noise variance $N_0 = 0.038\,\mathrm{V}^2$ and different levels of the power constraint P for a source with $ka = 0.38$ and concentric spherical receiver with $b = 2\ \mathrm{m}$.

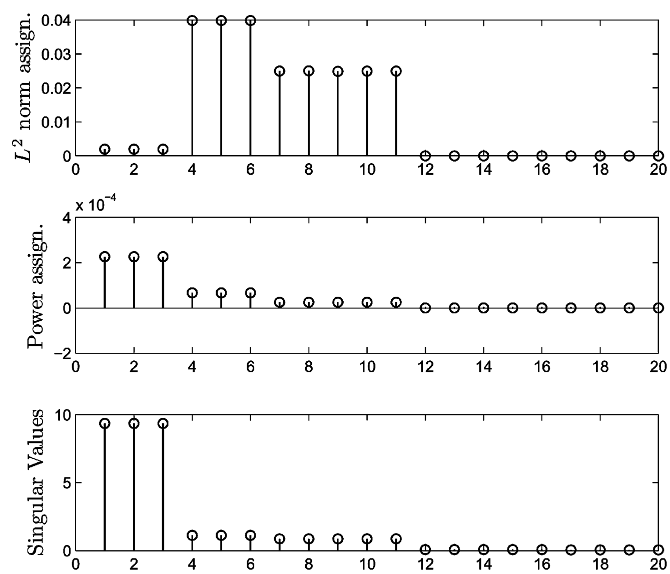

Fig. 3. Optimal L^2 norm and radiated power assignment as well as singular values corresponding to each channel for a source with $ka = 0.38$, $\mathcal{E} = 0.25\ \mathrm{A}^2/\mathrm{m}$, a radiated power constraint $P = 1\ \mathrm{mW}$, and a $\mathrm{SNR} = 10\ \mathrm{dB}$. The concentric spherical receiver was located at $b = 2\ \mathrm{m}$.

other cases, however, the singular values decay more gradually without a clear gap (e.g., for the case of small antennas or observation regions in the near zone), making it necessary to use regularization techniques which usually involve the noise level, signal energy, and other factors [41]. For example, Xu and Hanswamy [38] considered a definition of the NDF dependent of the noise levels and the source L^2 norm in the context of an electromagnetic communication system. In the present paper, we follow this more general philosophy of not truncating *a priori* the NDF, and instead let the very calculation of capacity reveal the NDF as function of the physically motivated constraints of power and source L^2 norm, the noise level, and the system geometry (including the near-field case).

2) Superdirectivity: For a given maximum allowed radiated power, constraining the L^2 norm of the source has the effect of controlling the level of superdirectivity or supergain of the antenna. In fact, a common metric of the superdirectivity is given by the geometric Q factor (see, e.g., [31], and similar factors used in antenna synthesis [59]–[61]) which is proportional to the ratio between source L^2 norm and radiated power.

For example, consider a source with $ka = 0.38$ and a concentric spherical receiver with $b = 2\ \mathrm{m}$. Fig. 1 shows the behavior of the capacity and NDF as a function of the ratio $\hat{Q} = \mathcal{E}/P$ for a fixed power $P = 1\ \mathrm{mW}$ and for different levels of the signal-to-noise ratio (SNR) defined as

$$\mathrm{SNR} = \frac{2\eta P}{N_0}. \tag{46}$$

The plot shows that capacity and NDF increase as the ratio $\hat{Q}$ (the source L^2 norm) increases, as expected, but also a plateau is eventually reached beyond which further increase is negligible. The same general behavior is illustrated from a complementary point of view in Fig. 2, where we consider different values of the radiated power constraint P and a fixed variance $N_0 = 0.038\ \mathrm{V}^2$ (for this variance the SNR is between 10 and 17 dB depending on the values of the power P).

It is interesting to note that the L^2 norm assignment is no longer as obvious as in the case with only the L^2 norm constraint where the channels with the larger gains (larger singular values) are always the optimal to use. Due to the interplay between power and source L^2 norm constraints this may no longer be the case. For example, Fig. 3 shows the L^2 norm assignment for the system in the previous example with $Q = 100$, $P = 1\ \mathrm{mW}$, and $\mathrm{SNR} = 10\ \mathrm{dB}$. In this case, channels with a lower gain are assigned most of the available L^2 norm while the channels with higher gain are assigned most of the radiated power.

3) Near- to Far-Field Transition: An important factor that affects the capacity and NDF is the distance between receiver

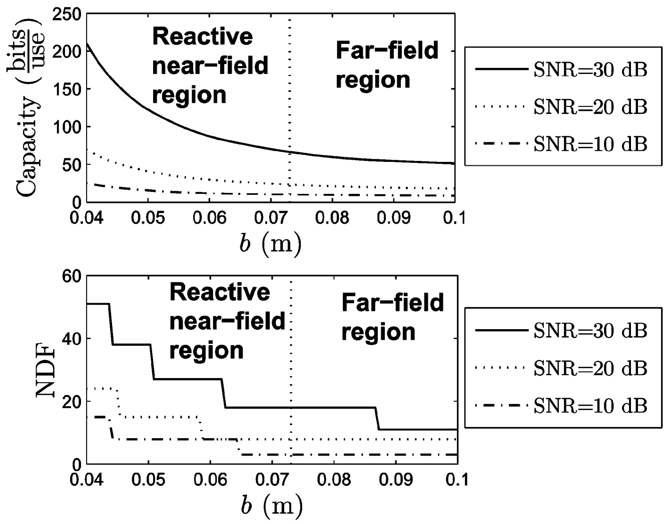

Fig. 4. Space capacity and NDF for a source of radius $a = 0.02$ m ($ka = 0.38$) as a function of the receiver radius b for several values of the SNR. Also shown in the plot are the two relevant field regions (see discussion in text).

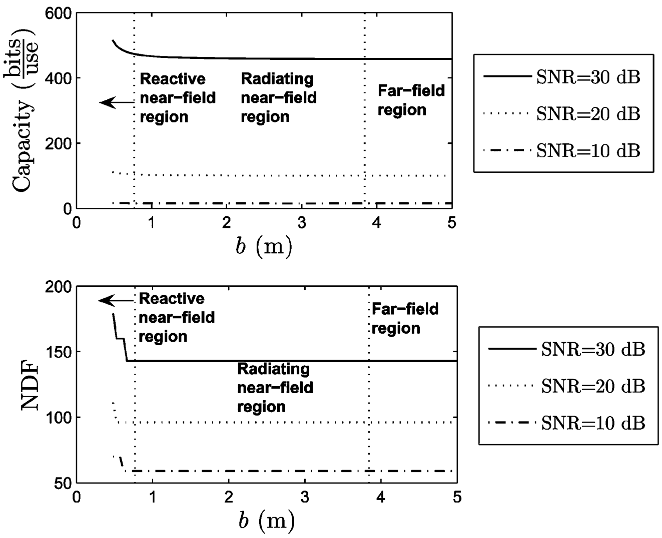

Fig. 5. Space capacity and NDF for a source of radius 0.4 m ($ka = 7.54$) as a function of the receiver radius b for several values of the SNR. Also shown in the plot are the three relevant field regions (see discussion in text).

and source. This behavior arises from the factor $||\mathbf{A}_{j,l,m}||^2_{\mathcal{Y}}$ of the singular values given by (19) and (20) which depends on the receiver radius b. The plot in Fig. 4 shows, for the same source with $ka = 0.38$, the capacity and the NDF corresponding as a function of the receiver distance b for several values of the ratio SNR. In the plot the outer boundary of the reactive near-field region is marked as a dotted line at a distance of $\lambda/2\pi$ from the surface of the antenna (a usual estimate for small radiators [62]) where λ is the wavelength of the radiation. Fig. 5 shows the same plot for a larger source ($a = 0.4$ m) with $ka = 7.54$ where the three relevant field regions have been indicated. The outer boundary of the reactive near-field region is taken as $0.62\sqrt{(2a)^3/\lambda}$ and the outer boundary of the radiating near-field region is taken as $2(2a)^2/\lambda$ [62]. For both sources, the NDF and capacity decrease rapidly, converging to the corresponding far-field values as the receiver distance b approximates the outer boundary of the reactive near-field region. In particular, in both cases the NDF converges to a minimum value as the receiver exits the reactive near-field region.

4) Receive Mode: It is important to mention that the previous developments also apply for a receiver antenna surrounded by a spherical transmitter if the medium is such that the reciprocity condition $(\mathbf{G}(\mathbf{r},\mathbf{r}') = \mathbf{G}(\mathbf{r}',\mathbf{r}))$ holds, which is the case for free space. In particular, it can be shown that for both situations (transmit versus receive) the singular values are the same implying that the capacity given by (45) also remains unchanged. This remark holds for all the capacity calculations of this paper.

D. Space Capacity With Radiated Power Constraint

Consider now having an unconstrained source L^2 norm, which can be shown to correspond to using $\lambda_1 = 0$ in (36). The capacity expression in (45) then reduces to

$$C = \sum_{j,l,m} \left[\log_2 \left(\frac{2\eta \left[\sigma_l^{(j)}\right]^2}{\left[\hat{\sigma}_l^{(j)}\right]^2 N_0 \lambda_2} \right) \right]^+ \left[\frac{\text{bits}}{\text{use}}\right] \quad (47)$$

where λ_2 satisfies

$$\sum_{j,l,m} \left(\frac{1}{\lambda_2} - \frac{\left[\hat{\sigma}_l^{(j)}\right]^2 N_0}{2\eta \left[\sigma_l^{(j)}\right]^2} \right)^+ \leq P. \quad (48)$$

Since the far-field singular values $\hat{\sigma}_l^{(j)}$ decrease more rapidly than the near-field singular values $\sigma_l^{(j)}$ the capacity in (47) will not converge in general. This unbounded behavior of the capacity due to unconstrained superdirectivity has been reported before in [28] and [31].

A special situation occurs when the receiver is in the far-field region where $\sigma_l^{(j)} = \hat{\sigma}_l^{(j)}$ so that the singular values in (47) and (48) cancel, yielding a capacity

$$C \approx \lim_{N\to\infty} N \log_2 \left(1 + \frac{2\eta P}{N_0 N}\right) = \frac{2\eta P}{N_0} \log_2 e \quad (49)$$

that is independent of the singular values $\sigma_l^{(j)}$ and that increases linearly with the SNR. This comes at the expense of a source with L^2 norm

$$\mathcal{E} = \lim_{N\to\infty} \sum_{j=1}^{2} \sum_{m=-l}^{l} \sum_{l=1}^{N} \frac{2\eta P}{\left[N(N+2)\hat{\sigma}_l^{(j)}\right]^2} \quad (50)$$

that is unbounded due to the exponentially decaying behavior of the singular values for large l [4], [43], [45], [63]. Note that the last result in (49) occurs because of the particular geometry of the receiver and that this may not be the case for more arbitrary configurations as illustrated, for example, in [28].

E. Space Capacity With Source L^2 Norm Constraint

Consider now the complementary situation where only the source L^2 norm is directly bounded, which corresponds to $\lambda_2 = 0$. The capacity expression in (45) reduces to

$$C = \sum_{j,l,m} \left[\log_2 \frac{\left[\sigma_l^{(j)}\right]^2}{N_0 \lambda_1} \right]^+ \tag{51}$$

where λ_1 satisfies

$$\sum_{j,l,m} \left(\frac{1}{\lambda_1} - \frac{N_0}{\left[\sigma_l^{(j)}\right]^2} \right)^+ \tag{52}$$

and can be computed via the well-known procedure called "water-filling" [20] or "water-pouring" [64].

IV. Space-Time Electromagnetic Information Capacity

In this section, we consider the practical scenario of (temporally) bandlimited signals motivated by the requirement of limiting the bandwidth of transmitted signals (to avoid interferences, follow regulations, and so on). It will be henceforth assumed that this bandwidth is within the inherent bandwidth limitations of practical antenna systems. However, we shall not dwell on particular antenna matching networks and the Bode-Fano matching theory that has already been treated in the antenna context in [32], [33], [65]. Our focus is the first principles source-to-field channel within a rather broad context pertinent to wideband radiating, propagating and even scattering systems.

In this section, the previous developments for time-harmonic sources operating at frequency f are extended to time-varying sources. To address this bandlimited electromagnetic channel case, we derive in the following the space-time generalization, within the particular electromagnetic physical framework, of the classical information theory for temporally bandlimited channels developed by Shannon [19] and elaborated in detail also by Gallager [39]. Methodologically, the particular approach adopted next borrows from the time domain treatment of bandlimited channels by Gallager [39]. The main ingredients of the theory are developed first for the particular case of a source that radiates during the finite time period $(-T/2, T/2)$ an electric field that the receiver measures at all times (see Fig. 6). The bandlimitation constraint is modeled by a bandpass filter of bandwidth W at the transmitter, that will be implemented *after* the companion time limitation consisting on masking the temporal transmit signal within the window $(-T/2, T/2)$. Later we consider the more general scenario of transmitter and receiver systems that are time limited such that the transmitter operates (is turned on) during a time window $(-T/2, T/2)$ of duration T, while the receiver operates (captures electromagnetic signals) during a time window (T_3, T_4). Given that in this theory the transmitter and receiver are both space and time limited (the spatial part is the finite size constraint), this is an information theory for space-limited, time-limited, and essentially bandlimited electromagnetic channels. Finally, the same developments for arbitrary T, T_3, and T_4, will naturally render the respective information capacity for bandlimited electromagnetic systems by means of a limiting procedure involving $T_3 = -T/2, T_4 = T/2, T \to \infty$.

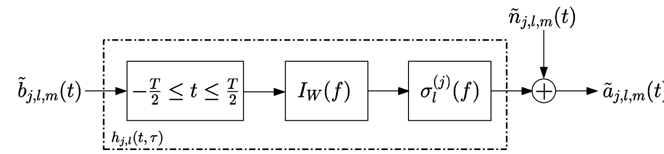

Fig. 6. Channel model with a transmission time limitation.

A. Spatio-Temporal Analysis

As in Section II, for each frequency f the source $\mathbf{J}(\mathbf{r}, f)$ can be represented in terms of a set of equivalent independent circuits [33], [34], [66] representing each of the spherical modes in the multipole expansion of the electric field in (9). This spherical expansion leads to an infinite set of waveforms of the form

$$a_{j,l,m}(f) = \sigma_{W,l}^{(j)}(f) b_{j,l,m}(f) \quad \forall\, j = 1, 2; \quad l = 1, 2, \ldots; \quad m = -l, -l+1, \ldots, l-1, l$$

where $\sigma_{W,l}^{(j)}(f) = \sigma_l^{(j)}(f) I_W(f)$, $\sigma_l^{(j)}(f)$ is given by (23), $I_W(f) = 1$ for f in the band of interest, $a_{j,l,m}(f)$ is defined in (6), and $b_{j,l,m}(f)$ is defined in (7).

Applying the inverse Fourier transform so that, for example, $\tilde{a}_{j,l,m}(t) = \int_{-\infty}^{\infty} a_{j,l,m}(f) e^{\imath 2\pi f t} df$ and considering the presence of white Gaussian noise at the receiver we find the model

$$\tilde{a}_{j,l,m}(t) = \int \tilde{b}_{j,l,m}(\tau) \tilde{\sigma}_{W,l}^{(j)}(t-\tau) d\tau + \tilde{n}_{j,l,m}(t) \quad \forall\, j = 1, 2; \quad l = 1, 2, \ldots; \quad m = -l, -l+1, \ldots, l-1, l; \quad t \in R \tag{53}$$

where each spatial model j, l, m represents a continuous-time random process (waveform). For simplicity, all random processes are assumed wide-sense stationary [67] so that the autocorrelation of the input signal $\tilde{b}_{j,l,m}(t)$ is $R_{\tilde{b}_{j,l,m}}(\tau) = E\left[\tilde{b}_{j,l,m}(t) \tilde{b}_{j,l,m}^*(t-\tau)\right]$ and the autocorrelation of the noise signal $\tilde{n}_{j,l,m}(t)$ is $R_{n_{j,l,m}}(\tau) = E\left[\tilde{n}_{j,l,m}(t) \tilde{n}_{j,l,m}^*(t-\tau)\right] = N_0 \delta(\tau)$. It is also assumed that the noise signals corresponding to different spatial modes are independent so that $E\left[\tilde{n}_{j,l,m}(t) \tilde{n}_{j',l',m'}^*(t-\tau)\right] = N_0 \delta(\tau) \delta_{j,j'} \delta_{l,l'} \delta_{m,m'}$. It can be shown from (17), (19), (20) and (23) that $\sigma_l^{(j)}(f)$ is a real and even function with respect to the frequency, so that the inverse Fourier transform is also real.

B. Space-Time Capacity With Source L^2 Norm Constraint

We will first consider the case of a single constraint in the source L^2 norm in detail. Then we will show the corresponding results for the case of a radiated power constraint and for the simultaneous constraining of the source L^2 norm and radiated power.

1) Finite Transmission Window: Consider a source that radiates only during a finite time period $-T/2 \le \tau \le T/2$ while the received signal is measured at all times. As shown in Fig. 6, this input time limitation can be modeled by defining the indicator function

$$I_{-T/2,T/2}(\tau)=\begin{cases}1 & \frac{-T}{2}\le\tau\le\frac{T}{2}\\ 0 & \text{otherwise}\end{cases}$$

and the auxiliary time-variant filter [39, Sec. 8.4]

$$h_{j,l}(t,\tau)=\tilde{\sigma}_{W,l}^{(j)}(t-\tau)I_{-T/2,T/2}(\tau). \quad (54)$$

The singular functions of the linear mapping associated to the filter $h_{j,l}(t,\tau)$ are found from the integral equations

$$\int \mathcal{H}_{j,l}(\tau_1,\tau_2)\phi_{j,l,m,k}(\tau_2)d\tau_2=\left[\lambda_{j,l,m,k}\right]^2\phi_{j,l,m,k}(\tau_1) \quad (55)$$

and

$$\int \phi_{j,l,m,k}(\tau)h_{j,l}(t,\tau)d\tau=\lambda_{j,l,m,k}\theta_{j,l,m,k}(t) \quad (56)$$

where $\phi_{j,l,m,k}(t)$ and $\theta_{j,l,m,k}(t)$ are the singular functions, $\lambda_{j,l,m,k}$ are the singular values, and

$$\mathcal{H}_{j,l}(\tau_1,\tau_2)=\int h_{j,l}(t,\tau_1)h_{j,l}(t,\tau_2)dt.$$

In view of (54)

$$\begin{aligned}\mathcal{H}_{j,l}(\tau_1,\tau_2)=&I_{-T/2,T/2}(\tau_1)I_{-T/2,T/2}(\tau_2)\\ &\times\int\tilde{\sigma}_{W,l}^{(j)}(t-\tau_1)\tilde{\sigma}_{W,l}^{(j)}(t-\tau_2)dt\\ =&I_{-T/2,T/2}(\tau_1)I_{-T/2,T/2}(\tau_2)\\ &\times\int\left[\sigma_{W,l}^{(j)}(f)\right]^2e^{\imath 2\pi f(\tau_1-\tau_2)}df. \quad (57)\end{aligned}$$

An important property of the singular values $\lambda_{j,l,m,k}$ and the singular functions $\phi_{j,l,m,k}(t)$ is that they are the solution of the maximization problem [39, Eq.(8.4.28)]

$$\left[\lambda_{j,l,m,k}\right]^2=\max_{\|x\|=1}\left\|\int h_{j,l}(t,\tau)x(\tau)d\tau\right\|^2 \quad (58)$$

while the singular functions $\theta_{j,l,m,k}(t)$ are the solution to

$$\left[\lambda_{j,l,m,k}\right]^2=\max_{\|y\|=1}\left\|\int h_{j,l}(t,\tau)y(t)dt\right\|^2. \quad (59)$$

The orthonormal functions $\phi_{j,l,m,k}$ are the set of time-limited functions [which follows from (55) and (57)] whose output to the propagation channel has the largest L^2 norm [following from (58)] and, hence, are essentially bandlimited. Therefore, they are the ideal expansion functions for the input signals $\tilde{b}_{j,l,m}(t)$. On the other hand, the orthonormal functions $\theta_{j,l,m,k}$ are the set of bandlimited functions [following from (56)] with the largest L^2 norm concentrated in the time interval $(-T/2,T/2)$ [following from (59)]. Thus, the received signal $\tilde{a}_{j,l,m}$ will be represented by means of this set of functions.

The expansion in terms of the singular functions $\phi_{j,l,m,k}$ and $\theta_{j,l,m,k}$ yields the discrete independent parallel Gaussian channels

$$\begin{aligned}a_{j,l,m,k}=&\lambda_{j,l,m,k}b_{j,l,m,k}+n_{j,l,m,k}\\ &\forall\, j=1,2; l=1,2,\ldots;\\ &m=-l,-l+1,\ldots,l-1; k=1,2,\ldots. \quad (60)\end{aligned}$$

where $a_{j,l,m,k}=(\theta_{j,l,m,k},\tilde{a}_{j,l,m})$, $b_{j,l,m,k}=(\phi_{j,l,m,k},\tilde{b}_{j,l,m})$, and $n_{j,l,m,k}=(\theta_{j,l,m,k},\tilde{n}_{j,l,m})$ and where $(f,g)=\int f^*(t)g(t)dt$.

The L^2 norm constraint of the source $\mathcal{J}(\mathbf{r},t)$ is given by

$$\sum_{j,l,m}\sum_{k=1}^{\infty}E\left[|b_{j,l,m,k}|^2\right]\le T\mathcal{E}. \quad (61)$$

Letting the capacity for a time interval T be denoted by C_T and following the procedure in Section III-E it is found that

$$C_T=\sum_{j,l,m}\sum_{k=1}^{\infty}\log_2\left(1+\frac{\left[\lambda_{j,l,m,k}\right]^2\hat{\mathcal{E}}_{j,l,m,k}}{N_0}\right) \quad (62)$$

where

$$\hat{\mathcal{E}}_{j,l,m,k}=\left(\nu-\frac{N_0}{\left[\lambda_{j,l,m,k}\right]^2}\right)^+ \quad (63)$$

with ν selected so that

$$\mathcal{E}=\frac{1}{T}\sum_{j,l,m}\sum_{k=1}^{\infty}\left(\nu-\frac{N_0}{\left[\lambda_{j,l,m,k}\right]^2}\right)^+. \quad (64)$$

In general, the singular values $\lambda_{j,l,m,k}$ are determined by solving the eigenvalue problem shown in (55). However, an important special case occurs if $\sigma_{W,l}^{(j)}$ is flat in the frequency band of interest. In this case the functions $\theta_{j,l,m,k}$ in (56) are essentially given by the prolate spheroidal wavefunctions [9], [39] (as can be shown by considering the lowpass equivalent of the system [68, App. B] so that the band of interest becomes $[-W/2,W/2]$) which are the bandlimited functions with the largest L^2 norm in the transmission time window. Furthermore, the singular values $\lambda_{j,l,m,k}$ are essentially a product of two gains: a spatial gain given by $\sigma_{W,l}^{(j)}$ (which depends on the spatial limitation in size of the transmitter and receiver) and a temporal gain associated to the prolate spheroidal wave functions (which is a function of the time-frequency product WT [9, p. 45]).

The NDF is obtained from the number of channels with nonzero functional energy $\hat{\mathcal{E}}_{j,l,m,k}$ in (63) and represents the number of discrete spatio–temporal modes that are relevant for the current geometry, constraints, and noise level. As an illustration consider a source with $ka=0.38$ operating at a central frequency $f=900$ MHz and a narrow bandwidth such that the quantities $\sigma_{W,l}^{(j)}$ are essentially constant (plot not shown) and the previous approximation is applicable. Fig. 7 shows the capacity in (62) and the NDF as a function of the SNR parameterized by the parameter $c_p=\pi WT/2$. As expected, increasing the time-bandwidth product (WT) increases the number of spatio–temporal degrees of freedom and the capacity.

These results can also shed light on the near to far field transition for time-domain fields. For example, consider the previous source with $W=25$ KHz, and a $\mathrm{SNR}=10$ dB. Fig. 8 shows the capacity and NDF as a function of the receiver distance b

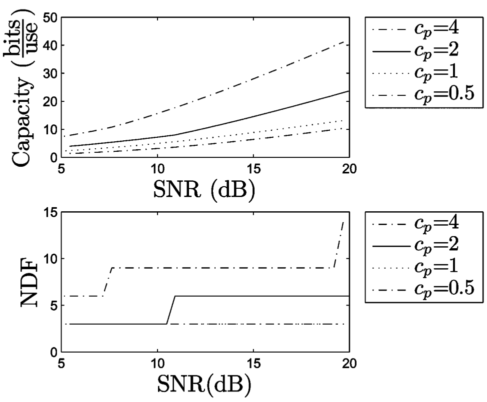

Fig. 7. Capacity and NDF versus SNR for different values of the parameter $c_p = \pi WT/2$ for a source with $ka = 0.38$ and a concentric spherical reception region of radius $b = 2$ m.

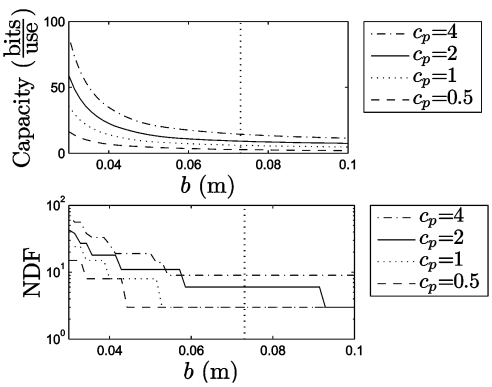

Fig. 8. Capacity and NDF versus receiver distance b for different values of the parameter $c_p = \pi WT/2$ for a source with $ka = 0.38$, $W = 25$ KHz, and a SNR $= 10$ dB.

for several values of the parameter c_p. The NDF converges to a minimum value (which is a sign of leaving the reactive near-field region in the time-harmonic case as shown in Figs. 4 and 5) at a shorter distance for smaller values of the parameter c_p which measures the time-bandwidth product. For reference, we also include the reactive region boundary for the central frequency of 900 MHz.

Another important aspect that can be illustrated by means of this analysis is the effect of the length of the transmission time T on the capacity per unit time C_T/T. Intuitively, having a larger transmission time should allow the construction of more efficient coding schemes. Fig. 9 shows, for a source with $ka = 0.38$ operating at a central frequency $f = 900$ MHz, a fixed and narrow bandwidth of 25 KHz, and a source L^2 norm constraint of $\mathcal{E} = 0.25\ \mathrm{A}^2/\mathrm{m}$ the quantity C_T/T as a function of the transmission time as measured by means of the time-bandwidth product $c_p \equiv \pi WT/2$. The plot shows that the capacity in bits/s increases with the transmission time T converging asymptotically to a constant. Also shown in the figure is the NDF which, for large c_p, increases almost linearly with the transmission time.

2) Finite Transmission and Reception Windows: Consider now limiting the transmission time to the finite time period $-T/2 \leq \tau \leq T/2$ while measuring the received signal for

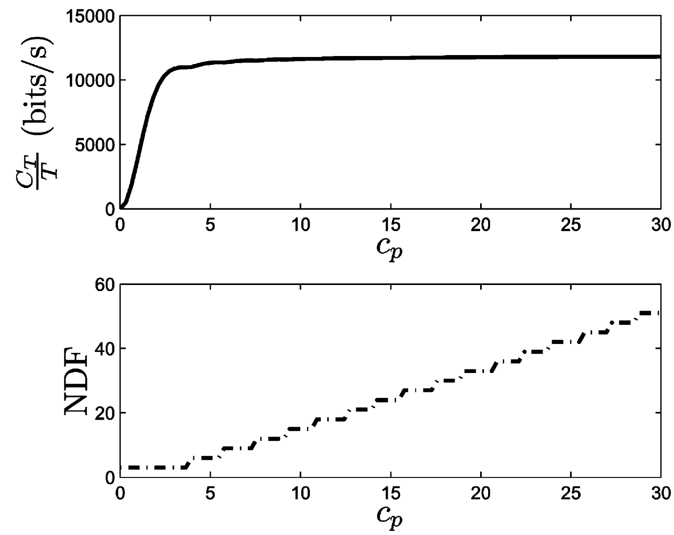

Fig. 9. Capacity in bits/s versus c_p for a source with $ka = 0.38$, bandwidth of 25 KHz, and a L^2 norm constraint $\mathcal{E} = 0.25\ \mathrm{A}^2/\mathrm{m}$.

the finite time period $T_3 \leq t \leq T_4$. As before this situation is modeled by means of an auxiliary filter given by

$$h_{j,l}(t,\tau) = I_{T_3,T_4}(t)\tilde{\sigma}_{W,l}^{(j)}(t-\tau)I_{-T/2,T/2}(\tau) \tag{65}$$

and the signals are expanded in terms of the singular system of the linear mapping associated to (65). In this case the singular functions ϕ and θ are both time-limited and essentially bandlimited. The capacity for this space-limited, time-limited, and essentially bandlimited system can be found from (62) and (63) where now $\lambda_{j,l,m,k}$ represents the singular values of the linear mapping associated to $h_{j,l}(t,\tau)$ in (65).

3) Bandlimited Channel: As shown in Fig. 9 the capacity per unit time converges to a constant value as the transmission time increases. This value can be found by means of the following procedure. In the previous development let $T_3 = -T/2$, $T_4 = T/2$, and $T \to \infty$ and define the capacity in bits/s as $C = \lim_{T\to\infty} 1/TC_T$ [19]. Then substituting (63) in (62) and applying the Toeplitz distribution theorem for continuous processes ([39, Lemma 8.5.7] or [64, Theorem C.3]) yields

$$C = \sum_{j,l,m} \int_W \left[\log_2\left(\nu\frac{\left[\sigma_l^{(j)}(f)\right]^2}{N_0}\right)\right]^+ df \left[\frac{\text{bits}}{\text{s}}\right] \tag{66}$$

where ν is selected so that

$$\mathcal{E} = \sum_{j,l,m} \int_W \left(\nu - \frac{N_0}{\left[\sigma_l^{(j)}(f)\right]^2}\right)^+ df \tag{67}$$

which are the spatio–temporal version of the well-known expressions for the capacity of bandlimited waveform channels with a constraint in source L^2 norm [compare, e.g., to [39, Eqs. (8.5.72), (8.5.73)] involving temporal channels and with [33, Eqs. (A.9), (A.10)] involving spatio–temporal channels in the framework of orthogonal frequency-division multiplexing (OFDM)].

C. Space-Time Capacity With Radiated Power Constraint

The radiated power constraint for a transmission time window T is given by

$$\frac{1}{2\eta}\sum_{j,l,m}\sum_{k=1}^{\infty}\left[\hat{\lambda}_{j,l,m,k}\right]^2 E\left[|b_{j,l,m,k}|^2\right] < TP \tag{68}$$

where $\hat{\lambda}_{j,l,m,k}$ are the singular values in (56) corresponding to a receiver located in the far zone.

Following a procedure analogous to that of the time-harmonic counterpart, Section III-D, it is found that the space-limited and time-limited capacity with a radiated power constraint is given by

$$C_T = \sum_{j,l,m}\sum_{k=1}^{\infty}\log_2\left(\nu\frac{2\eta\left[\lambda_{j,l,m,k}\right]^2}{\left[\hat{\lambda}_{j,l,m,k}\right]^2 N_0}\right)\left[\frac{\text{bits}}{\text{use}}\right] \tag{69}$$

where ν is chosen so that

$$\frac{1}{T}\sum_{j,l,m}\sum_{k=1}^{\infty}\left(\nu-\frac{\hat{\lambda}_{j,l,m,k}N_0}{2\eta\lambda_{j,l,m,k}}\right)^+ = P.$$

Letting $T\to\infty$ yields the space-limited and bandlimited capacity

$$C = \sum_{j,l,m}\int_W \log_2\left(\nu\frac{2\eta\left[\sigma_l^{(j)}(f)\right]^2}{\left[\hat{\sigma}_l^{(j)}(f)\right]^2 N_0}\right) df\left[\frac{\text{bits}}{\text{s}}\right] \tag{70}$$

where ν is chosen so that

$$\sum_{j,l,m}\int_W\left(\nu-\frac{\hat{\sigma}_l^{(j)}(f)N_0}{2\eta\sigma_l^{(j)}(f)}\right)^+ df = P.$$

Similarly to the time-harmonic case discussed in Section III-D, the capacity in (70) depends on the ratio between near- and far-field singular values. Since the far-field singular values decay faster this expression will not converge in general showing (again) that the radiated power constraint alone is not sufficient. However, for the particular case of a receiver in the far-field region the spatio–temporal channels become independent of the singular values $\sigma_l^{(j)}(f)$ and all channels are assigned the same power, yielding a capacity

$$C = \lim_{N\to\infty} NW\log_2\left(1+\frac{2\eta P}{N_0 WN}\right) = \frac{2\eta P}{N_0}\log_2(e). \tag{71}$$

This expression shows that the effect of the increase in the number of channels due to an unbounded source L^2 norm is to multiply the available bandwidth by the NDF so that in the limit as the NDF goes to infinity we obtain the well-known expression for the capacity of a channel with infinite bandwidth (see, e.g., [20, p. 250]). An approach to keep the L^2 norm of the source under control is to artificially limit the NDF to the estimate mentioned earlier in relation to (24), namely, $\text{NDF} = 2ka(ka+2)$, so that

$$C = 2ka(ka+2)\log_2\left(1+\frac{\eta P}{N_0 ka(ka+2)}\right). \tag{72}$$

An alternative approach is developed next wherein one considers the source L^2 norm and radiated power constraints simultaneously.

D. Space-Time Capacity With Multiple Constraints

For a source L^2 norm constraint of $\mathcal{E}$ and a power constraint of P the methods in Sections III-C and IV-B lead to the time-limited and space-limited capacity shown in (73) at the bottom of the page, where λ_1 and λ_2 are two nonnegative constants chosen to satisfy the constraints (61) and (68) and (37), (42), and (43) with the replacements $\sigma_l^{(j)} \to \lambda_{j,l,m,k}$, $\hat{\sigma}_l^{(j)} \to \hat{\lambda}_{j,l,m,k}$, $\mathcal{E}_{j,l,m} \to E[|b_{j,l,m,k}|^2]$, $\mathcal{E} \to T\mathcal{E}$, and $\sum_{j,l,m} \to \sum_{j,l,m}\sum_{k=1}^{\infty}$.

Letting $T\to\infty$ yields the bandlimited and space-limited capacity shown in (74) at the bottom of the page where λ_1 and

$$C_T = \sum_{j,l,m}\sum_{k=1}^{\infty}\left[\log_2\frac{\left[\lambda_{j,l,m,k}\right]^2}{N_0}\left(\frac{1}{\lambda_1+\lambda_2\frac{1}{2\eta}\left[\hat{\lambda}_{j,l,m,k}\right]^2}\right)\right]^+\left[\frac{\text{bits}}{\text{use}}\right] \tag{73}$$

$$C = \sum_{j,l,m}\int_W\left[\log_2\frac{\left[\sigma_l^{(j)}(f)\right]^2}{N_0}\left(\frac{1}{\lambda_1+\lambda_2\frac{1}{2\eta}\left[\hat{\sigma}_l^{(j)}(f)\right]^2}\right)\right]^+ df\left[\frac{\text{bits}}{\text{s}}\right] \tag{74}$$

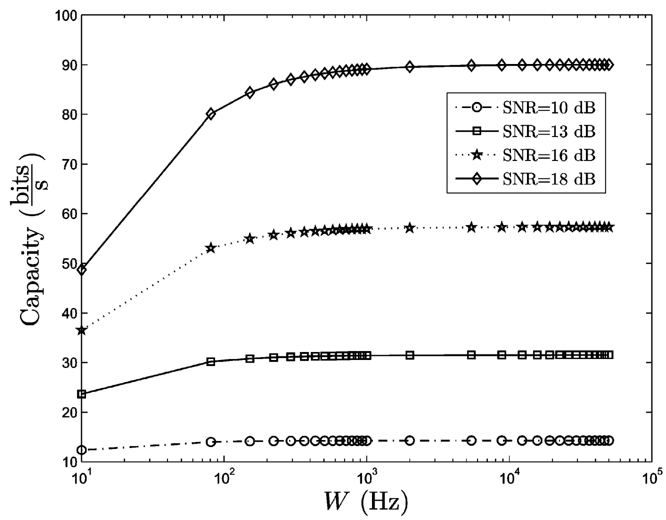

Fig. 10. Capacity in bits/s versus W for a source with $ka = 0.38$, $Q = 0.1$, and different values of the SNR.

λ_2 are two nonnegative constants chosen to satisfy (38) and (39) and (42) and (43) with

$$\mathcal{E}_{j,l,m}(f) = \left(\frac{1}{\lambda_1 + \lambda_2 \frac{1}{2\eta} \left[\hat{\sigma}_l^{(j)}(f) \right]^2} - \frac{N_0}{\left[\sigma_l^{(j)}(f) \right]^2} \right)^+ \tag{75}$$

and the replacements $\sum_{j,l,m} \rightarrow \sum_{j,l,m} \int_W$ and $\mathcal{E}_{j,l,m} \rightarrow \mathcal{E}_{j,l,m}(f)$. Expression (74) represents the spatio–temporal capacity for bandlimited channels with bandwidth W under constraints in source L^2 norm and radiated power. For example, Fig. 10 shows the capacity in bits/s versus the bandwidth W for a source with $ka = 0.38$, $Q = 0.1$, and different values of the SNR.

E. Broadband Versus Narrowband Information

We conclude this section by discussing how the more general time-domain theory of this section relates to the particular time-harmonic theory given earlier (Section III). For simplicity we will consider only the L^2 norm constraint in this section. In particular, if the bandwidth is sufficiently small so that $\sigma_l^{(j)}(f)$ is constant with respect to the frequency f then the spectral efficiency defined as [69]

$$\mathcal{C} = \frac{1}{W} C \, [\text{bits/s/Hz}] \tag{76}$$

is given by

$$\mathcal{C} = \sum_{j,l,m} \left[\log_2 \left(\nu \frac{\left[\sigma_l^{(j)}(f) \right]^2}{N_0} \right) \right]^+$$

which coincides with the spatial capacity in (51). The required condition for the constant ν in (67) becomes

$$\frac{\mathcal{E}}{W} = \sum_{j,l,m} \left(\nu - \frac{N_0}{\left[\sigma_l^{(j)}(f) \right]^2} \right)^+ .$$

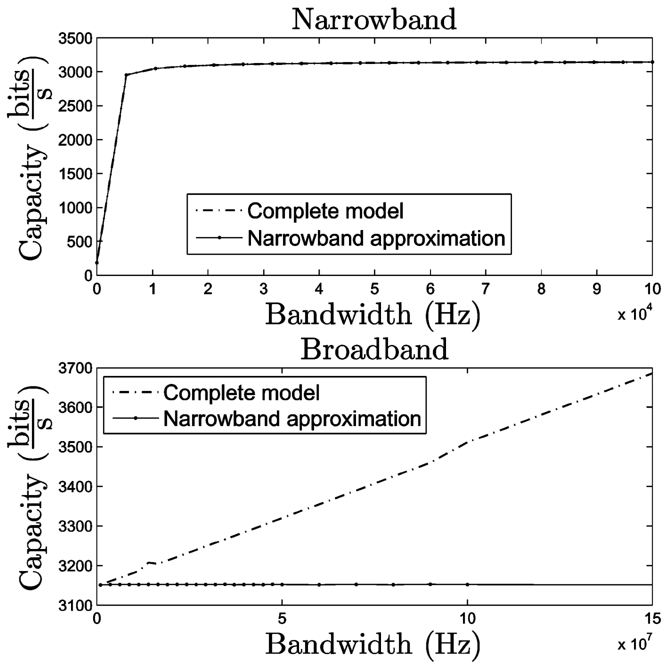

Fig. 11. Capacity in bits/s calculated using the space-time derivation in (51) and the narrowband theory in Section III-E versus the bandwidth W for a source with $ka = 0.38$ m and a receiver radius $b = 2$ m.

With the replacement $\mathcal{E}/W \rightarrow \mathcal{E}$ this equation also coincides with the condition in the spatial case given in (52). This shows that the developments in the time-harmonic case in Section III correspond to the special case of narrowband systems in the more general theory of this section.

This result also allows us to illustrate the importance of the full space-time analysis. As a motivating example consider two 25-KHz narrowband signals with a central frequency $f_1 = 800$ MHz and $f_2 = 900$ MHz, respectively, and a total available source L^2 norm of $\mathcal{E} = 0.25\ \mathrm{A}^2/\mathrm{m}$. One approach is to assign half the available L^2 norm to each signal and apply the narrowband theory of Section III to each signal independently. A better approach consists on treating both signals together by applying the general space-time theory of this section. In fact, numerical experiments showed that, for a noise variance $N_0 = 0.1\ \mathrm{V}^2$, by treating the signals independently the sum of both capacities was $C_1 + C_2 = 282.843$ bits/s while the capacity obtained by using the space-time theory was $C = 315.055$ bits/s. This result is expected since by using the space-time theory the resources are assigned optimally in *both* space and frequency. Fig. 11 further emphasizes the same point by showing a comparison between the capacity in bits/s calculated using the space-time derivation in (51) and the narrowband approximation above as a function of the bandwidth W for a source with $ka = 0.38$ and a central frequency of 900 MHz. While for small values of the bandwidth W both calculations are essentially equal (top plot), as W increases they diverge which shows the necessity of a broadband theory for larger bandwidths (bottom plot).

V. Conclusion

This paper investigated characterization of electromagnetic sources (antennas) via the information-theoretic concepts of Shannon's information capacity and NDF. The information capacity was calculated for a given additive Gaussian noise model considering different types of constraints: the well known bound on the source L^2 norm, a bound on the radiated power, and a novel combination of both source L^2 norm and radiated power bounds. Analytical expressions were given for the particular case of sources confined within a given spherical volume and receivers in free space that are supported in concentric spherical regions. These calculations are also valid for the reciprocal case of a receiver antenna contained in a spherical volume surrounded by a spherical transmitter as long as the medium is such that the reciprocity condition holds. This setup is of interest, for example, for testing and comparing antennas informationally. Section III considered the strictly spatial capacity and NDF calculations under a combined constraint of L^2 norm and radiated power of time-harmonic sources (although, it was shown in Section IV that this case also corresponds to narrowband sources). It was found (in Section III-D) that the power constraint alone is not necessarily well suited for practical estimates of capacity and NDF. Our analysis independently corroborates very recent work by Jensen and Wallace [28] on the same issue, and provides a conceptually simple and formally tractable *alternative* strategy (relative to the regularizing approach in [28]) to deal with this problem. In particular, our approach consists of constraining *also* the source L^2 norm, in order to get meaningful results. This provides a new unifying formulation to simultaneously incorporate the two most typical constraints found in practice (radiated power, and L^2 norm). These developments were then extended to more general time-dependent cases in Section IV where a spatio–temporal analysis was used in the calculation of the spatio–temporal capacity and NDF under source L^2 norm and radiated power constraints. There we developed a rigorous electromagnetic information theory for space-limited, time-limited, and essentially bandlimited electromagnetic systems, along with some of the resulting formal and physical insights and electromagnetic applications, as well as the associated calculation of capacity and NDF in space-limited and strictly bandlimited systems which were handled via an asymptotic procedure applied to the same time-limited theory. We hope that the present effort will stimulate further advances in the exciting field of electromagnetic information theory, with applications to wireless communications, information-theoretic characterization of antennas, and even information-theoretic assessment and novel algorithms relevant to imaging and scattering systems including the important broadband regime treated in this paper.

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